Indian Statistical Institute
Mid-Semestral Examination
Differential Topology: MMath II

Max Marks: 40 Time: 3 hours

(1) Let $G = \{(x, |x|) : x \in \mathbb{R}\}$. Show that G is the image of a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}^2$. [5]

(2) Let $X \subseteq \mathbb{R}^6$ be the subset defined by the equations

$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 1,$$
$$x_4^2 - x_5^2 - x_6^2 = -1.$$

Prove that X is a manifold. Find the dimension of X. [6]

(3) Let $f, g : S^1 \longrightarrow \mathbb{R}^3$ be smooth maps. Given $\varepsilon > 0$, show that there exists $v \in \mathbb{R}^3$ with $||v|| < \varepsilon$ such that

$$\{f(x) : x \in S^1\} \cap \{g(x) + v : x \in S^1\} = \emptyset.$$

[5]

(4) Define the notion of a Morse function on a manifold. Show that the critical points of a Morse function $f : \mathbb{R}^k \longrightarrow \mathbb{R}$ are isolated. Decide whether the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(x, y) = x^2 - 2xy + y^2$ is Morse. [8]

(5) Show that $O(n)$, the set of $n \times n$ orthogonal matrices, is a manifold. Describe $T_A(O(n))$ for $A \in O(n)$. [8]

(6) Let X, Z be submanifolds of Y. Prove that if $X \pitchfork Z$, then for all $y \in (X \cap Z)$ we have

$$T_y(X \cap Z) = T_y(X) \cap T_y(Z).$$

Is the converse true? [8]